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Sources for Kerr type space–times in general relativity

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Abstract. The source of a Kerr space–time is generally assumed to be reducible to a disc, although the singularity in the solution is in the form of a ring. This is because the maximally-extended exterior manifold is multiply-connected, being formed by two sheets joined through the ring. However, it is possible, in principle, to construct 'minimal' sources (reducible to the singularity) for space–times with such topology. This is illustrated in the Kerr case with zero mass constant, and some possibly general features are indicated.

Kerr's metric (Kerr 1963, Boyer and Price 1965, Carter 1966)

$$\begin{aligned} \mathrm{d}s^2 = \Sigma^{-1}\{(\Sigma-2mr)\,\mathrm{d}t^2-4amr\sin^2\theta\,\mathrm{d}t\,\mathrm{d}\phi-\sin^2\theta(\alpha\Sigma+2a^2mr\sin^2\theta)\,\mathrm{d}\phi^2\} \\ -\Sigma\,\mathrm{d}\theta^2-\alpha^{-1}\{4mr\,\mathrm{d}r\,\mathrm{d}t+4amr\sin^2\theta\,\mathrm{d}r\,\mathrm{d}\phi+\Sigma(1+2mr\alpha^{-1})\,\mathrm{d}r^2\} \end{aligned} \tag{1}$$

where $\alpha = r^2+a^2$, $\Sigma = r^2+a^2\cos^2\theta$, is generally believed to describe the exterior gravitational field due to a rotating massive body, the two constant parameters m, a, corresponding to mass and negative of angular momentum per unit mass, respectively. When $a = 0$ the metric reduces to Schwarzschild's solution, and when $m = 0$ each of the regions $r > 0$ and $r < 0$ is a complete flat space–time, the two being connected through the interior of a ring of radius a. The ring singularity ($r = 0$, $\cos\theta = 0$) is a general feature of Kerr's space–time and is a physical (ie irremovable) one. The metric possesses stationary axial symmetry, and each two dimensional meridian cross section $t =$ constant, $\phi =$ constant has the topology of a quadratic Riemann surface.

The problem of identifying the source of the field can be tackled in various ways, by examining the geodesic motion of particles or light rays (De Felice 1968), by a weak-field asymptotic approximation (Boyer and Price 1965), or by matching the Kerr metric to an explicit interior solution in a region containing the ring. If the source is assumed topologically spherical, the two branches of the exterior space are disjoint, and only one of them ($r > 0$) need be interpreted as physical space. This has the attraction of removing all problems associated with multivaluedness and the causality difficulties arising from the existence of closed time-like curves in the region $r < 0$. Hence the discovery of Kerr's metric has prompted many investigations of rotating fluid spheroids (Carter 1966), thin spheroidal massive shells (De la Cruz and Israel 1968), and discs (Lynden-Bell 1969 and Morgan 1969).

However, collapse processes render it important to determine the role of 'minimal' sources, that is, those which are reducible to the physical singularity in the exterior field. This note is intended to show that branching behaviour of the type encountered in Kerr's solution need not imply automatically that minimal sources cannot be constructed. Details for the Kerr metric will be published elsewhere when completed; here we shall merely illustrate one approach using the flat manifold ($m = 0$). Only the simplest possible form for the interior metric will be considered. Nevertheless, even here one can distinguish between features which are an obvious consequence of the absence of curvature in the exterior field and others which may well be more general.

We first construct the flat manifold afresh. Consider the Minkowski metric in cylindrical polar coordinates (R, ϕ, z) and time t, and introduce plane polar coordinates (R_1, ψ') in each meridian half-plane t = constant, ϕ = constant (taking $R = a$, $z = 0$ as pole, and the R direction for the polar line), by setting

$$R = R_1 \cos\psi' + a \qquad z = R_1 \sin\psi' \tag{2}$$

so that

$$ds^2 = dt^2 - dR_1^2 - R_1^2\, d\psi'^2 - (R_1 \cos\psi' + a)^2\, d\phi^2. \tag{3}$$

If (3) is interpreted on the manifold

$$M_1: \qquad -\infty < t < \infty \qquad 0 \leqslant R_1 \cos\psi' + a < \infty \qquad 0 < R_1 < \infty$$
$$0 \leqslant \phi \leqslant 2\pi \qquad 0 < \psi' < 2\pi \tag{4}$$

with identification of $\phi = 0$, $\phi = 2\pi$, we obtain a complete Minkowski space–time from which has been deleted (in each section t = constant) a ring ($R_1 = 0$) and its plane exterior. A second, identical, space–time is obtained by interpreting (3) on the manifold M_2, defined by (4) except that the last inequality is now replaced by $2\pi < \psi' < 4\pi$. Form a new manifold M as the union of M_1 and M_2 with the points $\psi' = 0, 2\pi, 4\pi$ adjoined, and identify $\psi' = 0$ with $\psi' = 4\pi$. Then M, with metric (3), has the required branching property about the points of the (excluded) singular ring, and so represents the specified exterior space–time.

It is convenient to write $\psi' = 2\psi$, so that (3) takes the form

$$ds^2 = dt^2 - dR_1^2 - A^2 R_1^2\, d\psi^2 - (R_1 \cos 2\psi + a)^2\, d\phi^2 \tag{5}$$

with $A = 2$ and $0 \leqslant \psi \leqslant 2\pi$. Note that for $A = 1$ we can adjoin the points $R_1 = 0$ without introducing a singularity there. Hence one method (many generalizations being evident) for constructing a nonsingular source to occupy a toroidal region $0 \leqslant R_1 \leqslant b$ ($b < a$) is to assume that (5) applies everywhere, with $A = A(R_1) > 0$, and

$$A = 2 \qquad R_1 \geqslant b$$
$$A = 1 \qquad R_1 = 0.$$

Provided that $A(R_1) \in C^3$ and has vanishing first derivative at $R_1 = 0$ (because of the polar nature of the coordinate R_1) the continuity conditions of Lichnerowicz will be satisfied throughout the space–time.

The field equations for (5) give

$$T_0^0 - T_i^i = 0 \qquad (i = 1, 2, 3) \tag{6}$$
$$T_1^1 = (8\pi R\beta^2)^{-1}(4R_1 - \beta\beta') \cos 2\psi \tag{7}$$
$$T_2^2 = 0 \tag{8}$$
$$T_3^3 = -(8\pi\beta)^{-1}\beta'' \tag{9}$$
$$T_2^1 = \beta^2 T_1^2 = (4\pi R\beta)^{-1}(R_1\beta' - \beta) \sin 2\psi \tag{10}$$

where $\beta = R_1 A(R_1)$, $\beta' = d\beta/dR_1$, $R = R_1 \cos 2\psi + a$, and the labelling of coordinates is

$x^0 = t$, $x^1 = R_1$, $x^2 = \psi$, $x^3 = \phi$. All other components of the stress–energy tensor T^μ_ν vanish. Equation (6) is an obvious consequence of the flatness of the exterior manifold, while (8) is probably due to the very restricted form assumed for the interior metric. The sign changes in T^1_1, T^2_1, T^1_2, as ψ varies, with R_1 fixed, are almost certainly attributable to the branching property, and therefore similar behaviour may be expected in minimal sources for a much wider class of space–times with this topology.

The boundary conditions on β are

$$R_1 = 0: \qquad \beta = 0 \qquad \beta' = 1$$

$$R_1 = b: \qquad \beta = 2b \qquad \beta' = 2 \qquad \beta'' = \beta''' = 0.$$

It follows that β'' must change sign at least once in $(0, b)$. Thus, if we picture the source as a set of concentric toroidal shells R_1 = constant, a consequence of (9) is that some of these shells will be in a state of high longitudinal compression and some in a state of high tension. This, too, is quite possibly typical for space–times with Kerr type branching, and should be verifiable without too much difficulty. The worst though not unexpected feature in the present case is, of course, the incompatibility of (6)–(9) with the physical requirement $T^0_0 > 0$.

The admissibility of minimal sources for Kerr space–times raises an important question. Under what conditions in the more familiar disc-like interiors will instability lead to separation of the matter, so that this complex situation can arise?

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